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COMMENT

Differential formulations of the renormalisation group in the large-n limit

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Abstract. We show that the large-n form of differential renormalisation group (RG) equations recently derived by Busiello *et al* from finite-difference recursion relations, can be obtained in a few simple steps by working from the start with the general form of the differential RG.

Busiello *et al* (1981, 1983) have recently derived differential formulations of the Wilson renormalisation group (RG) (Wilson and Kogut 1974) for the large-n limit of the classical n-vector model (Ma 1973). Busiello *et al* (1981, 1983) show that many of the standard fixed-point results of the original finite recursive formulations (Ma 1973, Szépfalusy and Tél 1979, 1980a, b) may be obtained in a more natural way from the differential renormalisation group (DRG) by applying standard techniques from the theory of quasi-linear partial differential equations. Other advantages of the DRG approach over the finite recursive form are well documented (Nicoll *et al* 1975, 1976, Nicoll and Chang 1978).

Busiello *et al* (1981, 1983) obtained the large-n form of the DRG simply by taking the differential limit of the recursion relations of Ma (1973) and Szépfalusy and Tél (1979, 1980a, b). In this comment we show that the large-n DRG equations can be obtained in a straightforward and self-contained manner by working from the start with a differential formulation of the RG (Wegner and Houghton 1973, Wilson and Kogut 1974, Nicoll and Chang 1978, Chang *et al* 1978 and Vvedensky *et al* 1983). Our approach will be seen to have the following advantages over that of Busiello *et al* (1981, 1983).

(i) We bypass completely the determination of recursion relations. The only step in our derivation which is specific to the large-n limit of the RG is the assignment of orders of magnitude to various quantities as $n \to \infty$.

(ii) Since we begin with an exact and closed-form formulation of the RG, we obtain limiting and approximate DRG equations as natural consequences of the exact equations (Nicoll *et al* 1976).

(iii) The basic functional form of the exact DRG generator fully exploits the formal similarity of all RG procedures for which the coarse-gaining is performed only in momentum space. Thus, any specification of the order parameter beyond the basic momentum dependence (vector or tensor components, time dependence, coupling to other fields) enters the coarse-gaining term of the DRG only as a trace over the associated field variable.

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We begin by considering an isotropic d-dimensional system ($d>2$) characterised by an n-component order parameter whose Fourier components we denote by $\psi_i(k)$, $i=1,\ldots,n$. Introducing the notation

$$x_{ij}(k,-k)\equiv\psi_i(k)\psi_j(-k), \qquad x(k,-k)\equiv\boldsymbol{\psi}(k)\cdot\boldsymbol{\psi}(-k),$$
$$x\equiv\frac{1}{2}\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^d}\,x(k,-k), \tag{1}$$

we suppose that the Hamiltonian $\mathscr{H}_G+\mathscr{H}$ is in the reduced form appropriate to the large-n limit (Ma 1973, Nicoll *et al* 1976):

$$\mathscr{H}_G=\frac{1}{2}\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^d}\,k^2x(k,-k), \qquad \mathscr{H}=\sum_{p=1}^{\infty}\frac{u_{2p}}{p!}x^p \tag{2}$$

where the u_{2p} are momentum independent. In our usual notation (Nicoll *et al* 1976, Vvedensky *et al* 1983), the DRG generator for $\mathscr{H}$ is

$$\frac{\partial\mathscr{H}}{\partial l}=\mathrm{d}\mathscr{H}+(2-d)x\frac{\partial\mathscr{H}}{\partial x}+\frac{1}{2}\int\frac{\mathrm{d}\Omega}{(2\pi)^d}\,\mathrm{Tr}\ln\left[\delta_{ij}\left(1+\frac{\partial\mathscr{H}}{\partial x}\right)+\frac{\partial^2\mathscr{H}}{\partial x^2}\,x_{ij}(q,-q)\right]$$

where in the case of momentum-independent $\mathscr{H}$ we have $\partial\mathscr{H}_G/\partial l=0$ and accordingly we have set $\eta=0$ (Nicoll *et al* 1976).

According to the central limit theorem, as $n\to\infty$ we have that $x(k,-k)$ and x are $\mathrm{O}(n)$ and we suppose that $u_{2p}=\mathrm{O}(n^{1-p})$ (Ma 1973), so that $\mathscr{H}=\mathrm{O}(n)$. On the other hand, the quantity $t\equiv\partial\mathscr{H}/\partial x$ is $\mathrm{O}(1)$ and has the same parameter space as $\mathscr{H}$. We may obtain the DRG generator for t by first differentiating (3) with respect to x,

$$\begin{aligned}\frac{\partial t}{\partial l}&=2t+(2-d)x\frac{\partial t}{\partial x}+\frac{1}{2}\int\frac{\mathrm{d}\Omega}{(2\pi)^d}\,\mathrm{Tr}\left\{\left[\delta_{ij}(1+t)+\frac{\partial t}{\partial x}\,x_{ij}(q,-q)\right]^{-1}\right.\\&\qquad\left.\times\left[\delta_{ij}\frac{\partial t}{\partial x}+\frac{\partial^2 t}{\partial x^2}\,x_{ij}(q,-q)\right]\right\}\\&=2t+(2-d)x\frac{\partial t}{\partial x}+\frac{1}{2}\int\frac{\mathrm{d}\Omega}{(2\pi)^d}\,\frac{n}{1+t}\frac{\partial t}{\partial x}+\mathrm{O}\left(\frac{\partial t}{\partial x}\right)\end{aligned} \tag{4}$$

and performing the variable change

$$x\to z=2\frac{(2\pi)^d}{S_d}(d-2)\frac{x}{n} \tag{5}$$

where S_d is the surface area of a unit d-sphere. Since the n-dependence in (4) is now explicit, we may take the limit $n\to\infty$

$$\frac{\partial t}{\partial l}=2t+(2-d)\left(z-\frac{1}{1+t}\right)\frac{\partial t}{\partial z} \tag{6}$$

which is the DRG equation derived by Busiello *et al* (1981).

For the large-n limit of critical dynamics, we consider a system again characterised by an n-component order parameter $\psi_i(xt)$, $i=1,\ldots,n$ whose dynamics are governed by the generalised Langevin equations

$$\dot{\psi}_i(x,t)=f_i[\psi(x,t),x,t]+\eta_i(x,t) \tag{7}$$

where the f_i are deterministic forces given in terms of the Hamiltonian by

$$f_i[\psi(x,t),x,t] = -\Gamma(x,t)\partial\mathcal{H}/\delta\psi_i(x,t) \tag{8}$$

and the η_i are stochastic forces which are assumed to have zero mean and to be uncorrelated in the sense that

$$\langle\eta_i(x,t)\eta_j(x',t')\rangle = 2\Gamma(x,t)\delta_{ij}\delta(x-x')\delta(t-t') \tag{9}$$

and Γ is a transport coefficient to be specified below. We Fourier transform in space and time and introduce the Fourier components of the order parameter ψ_i and the field ϕ_i conjugate to the noises through the notation $\psi_1^i(k\omega) \equiv \psi_i(k\omega)$ and $\psi_2^i(k\omega) \equiv \phi_i(k\omega)$. Introducing the variables,

$$\begin{aligned} x_{\alpha\beta}^{ij}(k\omega;-k,-\omega) &\equiv \psi_\alpha^i(k\omega)\psi_\beta^j(-k,-\omega), \\ x_{\alpha\beta}(k\omega;-k,-\omega) &\equiv \boldsymbol{\psi}_\alpha(k\omega)\cdot\boldsymbol{\psi}_\beta(-k,-\omega) \\ x_{\alpha\beta} &= \frac{1}{2}\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^d}\int\frac{\mathrm{d}\omega}{2\pi}\,x_{\alpha\beta}(k\omega;-k,-\omega), \end{aligned} \tag{10}$$

we suppose that the action $A_G + A$ is in the reduced form appropriate to the large-n limit (Szépfalusy and Tél 1979, 1980a, b):

$$\begin{aligned} A_G &= \frac{1}{2}\sum_{\alpha\beta}\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^d}\int\frac{\mathrm{d}\omega}{2\pi}\,r_{\alpha\beta}(k\omega)x_{\alpha\beta}(k\omega;-k,-\omega) \\ A &= \sum_{p=1}^{\infty}\sum_{q=1}^{p} u_{2p,2q}x_{12}^q x_{11}^{p-q} \end{aligned} \tag{11}$$

with

$$r_{11}=0,\qquad r_{12}=\mathrm{i}(k^2+\mathrm{i}\omega/\Gamma_k),\qquad r_{21}=\mathrm{i}(k^2-\mathrm{i}\omega/\Gamma_k),\qquad r_{22}=-2/\Gamma_k \tag{12}$$

and where the $u_{2p,2q}$ are momentum independent and we take $\Gamma_k = k^2$ (resp., 1) if the order parameter is conserved (resp., not conserved).

The DRG generator for A is, again in our usual notation (Chang *et al* 1978, Vvedensky *et al* 1983),

$$\begin{aligned} \frac{\partial A}{\partial l} &= (d+z)A + (2-d)x_{11}(\partial A/\partial x_{11}) + (2-d-z)x_{12}(\partial A/\partial x_{12}) \\ &\quad + \frac{1}{2}\int\frac{\mathrm{d}\Omega}{(2\pi)^d}\,\mathrm{Tr}\ln\bigg\{\delta_{ij}\bigg[r_{\alpha\beta}(q\omega)+\frac{\partial A}{\partial x_{\alpha\beta}}\bigg] \\ &\quad + \sum_{\alpha'\beta'}\frac{\partial^2 A}{\partial x_{\alpha\alpha'}\,\partial x_{\beta\beta'}}\,x_{\alpha'\beta'}^{ij}(q\omega;-q,-\omega)\bigg\} \end{aligned} \tag{13}$$

where in the case of momentum and frequency-independent A we have $\partial A_G/\partial l = 0$ and accordingly we set $\eta = 0$ and $z = 4$ (resp., 2) if the order parameter is conserved (resp., not conserved).

Since in the large-n limit $x_{\alpha\beta}(k\omega;-k,-\omega) = \mathrm{O}(n)$, $x_{\alpha\beta} = \mathrm{O}(n)$, and $u_{2p,2q} = \mathrm{O}(n^{1-p})$ (Szépfalusy and Tél 1979, 1980a), then $A = \mathrm{O}(n)$ as $n \to \infty$. Alternatively, the quantities $t_1 \equiv \partial A/\partial x_{12} = \partial A/\partial x_{21}$ and $t_2 \equiv \partial A/\partial x_{11}$ are each O(1) and *together* span the parameter space of A. We may obtain coupled DRG equations for the t_i by following steps analogous to (4)–(6). Differentiating (13) accordingly, we obtain for $i=1,2$

$$\frac{\partial t_i}{\partial l}=\lambda_i t_i+(2-d)x_{11}\frac{\partial t_i}{\partial x_{11}}+(2-d-z)x_{12}\frac{\partial t_i}{\partial x_{12}}$$
$$+\frac{1}{2}\int\frac{\mathrm{d}\Omega}{(2\pi)^d}\,\mathrm{Tr}\left\{\left[\delta_{ij}\left(r_{\alpha\beta}+\frac{\partial A}{\partial x_{\alpha\beta}}\right)+\sum_{\alpha'\beta'}\frac{\partial^2 A}{\partial x_{\alpha\alpha'}\,\partial_{\beta\beta'}}x^{ij}_{\alpha'\beta'}\right]^{-1}\right.$$
$$\left.\times\left[\delta_{ij}\frac{\partial t_i}{\partial x_{\alpha\beta}}+\sum_{\alpha'\beta'}\frac{\partial^2 t_i}{\partial x_{\alpha\alpha'}\,\partial x_{\beta\beta'}}x^{ij}_{\alpha'\beta'}\right]\right\} \tag{14}$$

with $\lambda_1=2$, $\lambda_2=2+z$. Then, defining

$$D=\det\left(r_{\alpha\beta}+\frac{\partial A}{\partial x_{\alpha\beta}}\right)=2t_2-(1+t_1)^2-\omega^2 \tag{15}$$

we expand (14) in analogy with (4):

$$\frac{\partial t_i}{\partial l}=\lambda_i t_i+(2-d)x_{11}\frac{\partial t_i}{\partial x_{11}}+(2-d-z)x_{12}\frac{\partial t_i}{\partial x_{12}}$$
$$+\int\frac{\mathrm{d}\Omega}{(2\pi)^d}\int\frac{\mathrm{d}\omega}{2\pi}\frac{n}{D}\left[\frac{\partial t_i}{\partial x_{11}}-(1+t_1)\frac{\partial t_i}{\partial x_{12}}\right]+\mathrm{O}\left(\frac{\partial t_i}{\partial x_{\alpha\beta}}\right). \tag{16}$$

Performing the ω and Ω integrations, making the variable changes

$$x_{12}\to x=2\frac{(2\pi)^d}{S_d}(d+z-2)\frac{x_{12}}{n},\qquad x_{11}\to y=2\frac{(2\pi)^d}{S_d}(d-2)\frac{x_{11}}{n} \tag{17}$$

and taking the limit $n\to\infty$, we obtain

$$\frac{\partial t_i}{\partial l}=\lambda_i t_i+(2-d-z)[x+F(t_1,t_2)]\frac{\partial t_i}{\partial x}+(2-d)[y-G(t_1,t_2)]\frac{\partial t_i}{\partial y} \tag{18}$$

where

$$F(t_1,t_2)=(1+t_1)G(t_1,t_2),\qquad G(t_1,t_2)=[(1+t_1)^2-2t_2]^{-1/2} \tag{19}$$

which to within an additive constant of x are the equations obtained by Busiello *et al* (1983). This additive constant represents the effect of causality in the path integral representation of the equations (7)–(9) (Bausch *et al* 1976) and may be eliminated by a simple variable transformation.

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